Modified Braid Equations for $SO_q(3)$ and noncommutative spaces

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Abstract

General solutions of the $\hat{R}TT$ equation with a maximal number of free parameters in the specrtal decomposition of vector $SO_q(3)$ $\hat{R}$ matrices are implemented to construct modified braid equations (MBE). These matrices conserve the given, standard, group relations of the nine elements of T, but are not constrained to satisfy the standard braid equation (BE). Apart from q and a normalisation factor our $\hat{R}$ contains two free parameters, instead of only one such parameter for deformed unitary algebras studied in a previous paper [1] where the nonzero right hand side of the MBE had a linear term proprotional to $(\hat{R}_{(12)} - \hat{R}_{(23)})$. In the present case the r.h.s. is, in general, nonliear. Several particular solutions are given (Sec.2) and the general structure is analysed (App.A). Our formulation of the problem in terms of projectors yield also two new solutions of standard (nonmodified) braid equation (Sec.2) which are further discussed (App.B). The noncommutative 3-spaces obtained by implementing such generalized $\hat{R}$ matrices are studied (Sec.3). The role of coboundary $\hat{R}$ matrices (not satisfying the standard BE) is explored. The MBE and Baxterization are presented as complementary facets of the same basic construction, namely, the general solution of $\hat{R}TT$ equation (Sec.4). A new solution is presented in this context. As a simple but remarkable particular case a nontrivial solution of BE is obtained (App.B) for $q = 1$. This solution has no free parameter and is not obtainable by twisting the identity matrix. In the concluding remarks (Sec.5), among other points, generalisation of our results to $SO_q(N)$ is discussed.

CPHT-S 011.0301

*Laboratoire Propre du CNRS UPR A.0014

1 Introduction

In a previous paper [1] a particular class of inhomogeneous, modified, braid equations (MBE) was shown to correspond to general solutions of $\hat{R}TT$ relations. Fundamental 2×2 T matrices and the corresponding 4×4 $\hat{R}(=PR)$ matrices for $GL_{p,q}(2)$, $GL_{g,h}(2)$ and $GL_{q,h}(1/1)$ were used as examples. The inputs were the known group relations of the elements (a,b,c,d) for each of the above cases. Then the most general solution (without imposing the Yang Baxter equation for R or, equivalently, the braid equation for $\hat{R}$) was sought, for each case, of the relation

$$\hat{R}T_1T_2 = T_1T_2\hat{R} \tag{1.1}$$

where

$$T_1 = T\otimes I_2, \quad T_2 = I_2\otimes T$$

The only constraint on $\hat{R}$ was the conservation of the given group relations for (a,b,c,d). In each case $\hat{R}$ was found to depend, , apart from the two parameters $((p,q),(g,h),(q,h))$ linearly on a third one (K) such that, for a suitable normalisation, one obtains

$$\hat{R}_{(12)}\hat{R}_{(23)}\hat{R}_{(12)} - \hat{R}_{(23)}\hat{R}_{(12)}\hat{R}_{(23)} = \Big(\frac{K}{K_1}-1\Big)\Big(\frac{K}{K_2}-1\Big)\Big(\hat{R}_{(23)}-\hat{R}_{(12)}\Big) \tag{1.2}$$

This is the MBE with

$$(K_1,K_2) = (1,p/q), \quad (1,1), \quad (1,1/q) \tag{1.3}$$

respectively for the above-mentioned cases.

It was pointed out in [1] that (1.2) reexpressed in terms of R, provides a particular, interesting class of modified quantum YB equations ($MQYBE$) of Gerstenhaber,Giaquinto and Schack (see [2] and sources cited therein).

The special features of (1.2), as indicated in [1] are as follows.

(1): The explicit stucture on the right carries interesting information. After obtaining the general solution of (1.1) and the MBE it corresponds to one obtains the unmodified,standard braid or YB matrices as byproducts. One just sets $K=K_1$ or $K=K_2$, the two solutions being related through

$$\hat{R}(K_2) = \hat{R}(K_1)^{-1} \tag{1.4}$$

(2): Setting, as explained in [1],

$$K = 2K_1K_2(K_1+K_2)^{-1} \tag{1.5}$$

one obtains

$$(\hat{R}(K))^2 = I \tag{1.6}$$

Hence the construction of "triangular" (or "unitary" or "coboundary") R matrices is again reduced to the choice of a particular value of K in the general solution. This aspect will be studied further below.

(3): It was pointed out in [1] that MBE and Baxterization are complementary facets of the same basic construction , namely, the general solution of (1.1). This links MBE to integrable models. This aspect will be taken up in $Sec.$4 with new examples.

(4): The parameter K plays an interesting role in noncommutative spaces obtained by implementing the general $\hat{R}(K)$. The detailed discussion in [1] of this aspect will be generalized to higher dimensions in $Sec.$(3).

Spectral decomposition and generalization to higher dimensions:

Our construction can be generalized to higher dimensional cases most conveniently by introducing arbitrary constant coefficients in the spectral decomposition of R matrices for *vector representations.* The standard solutions (not "modified" in our sense) are well-known [3, 4]. Instead of wading through larger and large number of group relations ($n^2(n^2-1)/2$ for n^2 elements of T) one starts with the following results for vector representations.

For $GL_q(N)$ one has, in terms of the projectors $P^{(\pm)}$,

$$\hat{R} = qP^{(+)} - q^{-1}P^{(-)} \tag{1.7}$$

where

$$P^{(+)}P^{(-)} = 0, \qquad (P^{(\pm)})^2 = P^{(\pm)}, \qquad P^{(+)} + P^{(-)} = I \tag{1.8}$$

Here $\hat{R}$ satisfies the braid equation and

$$(\hat{R} - qI)(\hat{R} + q^{-1}I) = 0 \tag{1.9}$$

One has also

$$P^{(+)} = \frac{(\hat{R} + q^{-1}I)}{(q + q^{-1})}, \qquad P^{(-)} = -\frac{(\hat{R} - qI)}{(q + q^{-1})} \tag{1.10}$$

If one sets, *with the same projectors*,

$$\hat{R}(u, v) = uP^{(+)} + vP^{(-)} \tag{1.11}$$

where (u, v) are non-zero, unequal but otherwise arbitrary parameters, one obtains

$$(\hat{R}(u, v) - uI)(\hat{R}(u, v) - vI) = 0 \tag{1.12}$$

$$P^{(+)} = \frac{(\hat{R}(u, v) - vI)}{(u - v)}, \qquad P^{(-)} = \frac{(\hat{R}(u, v) - uI)}{(v - u)} \tag{1.13}$$

Of the two parameters (u, v) one can be fixed by choosing a suitable normalisation, leading effectively to one independent, arbitrary parameter. Apart form differences in notations our construction of $\hat{R}(K)$ in [1] (see Sec.3.2 in [1]), namely

$$\hat{R}(K) = (1 - (K/K_1 + K/K_2))P_1 + P_2 \tag{1.14}$$

corresponds directly to (1.11) above. This and certain other aspects of our previous formalism *can be directly generalised* to $GL_q(N)$, though the noncommutative spaces will now be of N dimensions.

For $SO_q(N)$ (and for $Sp_q(N)$ which we do not consider here) there is a major change. One has now *three* projectors in the spectral decomposition of $\hat{R}$ matrices for vector representations. The consequences for MBE will be seen to be important.

For $\hat{R}$ satisfying the braid equation one obtains ([3], [4])

$$(\hat{R} - qI)(\hat{R} + q^{-1}I)(\hat{R} - q^{1-N}I) = 0 \tag{1.15}$$

and

$$\hat{R} = qP^{(+)} - q^{-1}P^{(-)} + q^{1-N}P^{(0)} \tag{1.16}$$

where (with (i, j) denoting a pair from $(+, -, 0)$)

$$P^{(i)}P^{(j)} = P^{(i)}\delta_{ij}, \qquad P^{(+)} + P^{(-)} + P^{(0)} = I \tag{1.17}$$

Generalising as before we introduce (with non-zero and unequal (u, v, w) and the projectors being the *same* as before, independent of (u, v, w))

$$\hat{R}(u, v, w) = uP^{(+)} + vP^{(-)} + wP^{(0)} \tag{1.18}$$

Now (denoting $\hat{R}(u, v, w)$ as $\hat{R}$),

$$(\hat{R} - uI)(\hat{R} - vI)(\hat{R} - wI) = 0 \tag{1.19}$$

and

$$\begin{aligned} P^{(+)} &= \frac{(\hat{R} - vI)(\hat{R} - wI)}{(u - v)(u - w)} \\ P^{(-)} &= \frac{(\hat{R} - uI)(\hat{R} - wI)}{(v - u)(v - w)} \\ P^{(0)} &= \frac{(\hat{R} - uI)(\hat{R} - vI)}{(w - u)(w - v)} \end{aligned} \tag{1.20}$$

Here, fixing the normalisation, *two* independent parameters are left.

In the next section we will study the MBE corresponding to (1.18). We will set $N = 3$. This will permit us to display explicitly matrices of managaeble size. *The essential new features will, however, be present already for* $N = 3$.

Let us now note how the number of coboundary (or unitary) solutions for vector representations changes with the number of projectors.

For $GL_q(N)$ it is seen from (1.8) and (1.11) that

$$\left(\hat{R}(u, v)\right)^2 = u^2P^{(+)} + v^2P^{(-)} = P^{(+)} + P^{(-)} \tag{1.21}$$

for

$$u^2 = v^2 = 1.$$

Hence, apart from an overall $(\pm)$ sign, the only nontrivial solution is

$$\hat{R}_c = P^{(+)} - P^{(-)} = I - 2P^{(-)} = -(I - 2P^{(+)}); \qquad \hat{R}_c^2 = I \tag{1.22}$$

For $SO_q(n)$, from (1.17) and (1.18) apart from an overall sign one obtains analogously *three* solutions

$$\hat{R}_c = (I - 2P^{(+)}); \quad (I - 2P^{(-)}); \quad (I - 2P^{(0)}) \tag{1.23}$$

Each satisfies

$$\hat{R}_c^2 = I$$

and the product of any two gives the third one with a change of sign. Thus, for example

$$(I - 2P^{(+)})(I - 2P^{(-)}) = -(I - 2P^{(0)}) \tag{1.24}$$

If the coefficient (-2) in (1.23) is replaced by an arbitrary number $\hat{R}$ still satifies as is easily seen a *quadratic* equation, not the cubic (1.19).

If complex solutions are considered for real q but with complex coefficients in (1.18) , one obtains the unitarity relation when (u, v, w) are phases in (1.18). Thus (with real deltas)

$$\hat{R} = e^{i\delta_1} P^{(+)} + e^{i\delta_2} P^{(-)} + e^{i\delta_3} P^{(0)} \tag{1.25}$$

gives, since the projectors are symmetric for the orhogonal case,

$$\hat{R}^\dagger \hat{R} = I$$

In (1.18), (u, v, w) were postulated to be unequal. This permits one to express *all the three* projectors in terms of $\hat{R}$ as in (1.20). But this is not obligatory. As noted in (1.23) other cases (with $u = v = -w = 1$ and so on) can indeed be of special interest. For (1.23) in each case one has *two* mutually orthogonal combinations. Selecting the second case, for example, one obtains

$$P^{(-)} = -\frac{1}{2}(\hat{R}_c - I), \qquad P^{(+)} + P^{(0)} = \frac{1}{2}(\hat{R}_c + I) \tag{1.26}$$

We conclude with a fully explicit statement of our approach. If one has

$$\hat{R} T_1 T_2 = T_1 T_2 \hat{R}$$

then any function $f(\hat{R})$ of $\hat{R}$ satisfies

$$f(\hat{R}) T_1 T_2 = T_1 T_2 f(\hat{R}) \tag{1.27}$$

To start with let us suppose (for definiteness, such a starting point not being essential) that $\hat{R}$ satifies BE.

For $GL_q(N)$, $\hat{R}$ satisfies a quadratic constraint (1.9). Hence any power series in $\hat{R}$ can be reduced to a linear function in $\hat{R}$. Hence, apart from an overall normalisation factor, the most general solution of (1.27), for a given set of group relations, becomes effectively (1.14) as studied in [1]. (It is easy to see from (1.7) and (1.10) that even fractional powers of $\hat{R}$ can be obtained as a linear funtion of $\hat{R}$ but, in general, with complex coefficients. Having noted this, we will usually implicitly consider real coefficients. An analogous situation will hold for the orthogonal case considered below. Complex coefficients, such as in (1.25) will not be introduced explicitly. Except when roots of unity are involved, complexification of our formalism is however straightforward.)

For $SO_q(N)$, $\hat{R}$ satisfies the a cubic constraint (1.15). Hence the general solution, using analogous arguments, is seen effectively to be, with constant coefficients c_i,

$$f(\hat{R}) = c_1(\hat{R})^2 + c_2\hat{R} + c_3 I \tag{1.28}$$

Using the spectral decomposition (1.16) along with (1.17),

$$\begin{gathered} f(\hat{R}) = c_1(q^2P^{(+)} + q^{-2}P^{(-)} + q^{2(1-N)}P(0)) \\ +c_2(qP^{(+)} - q^{-1}P^{(-)} + q^{(1-N)}P(0)) \\ +c_3(P^{(+)} + P^{(-)} + P^{(0)}) \end{gathered}$$

Hence, collecting together the coefficients one obtains the form

$$f(\hat{R}) = uP^{(+)} + vP^{(-)} + wP^{(0)} \tag{1.29}$$

This is the motivation for (1.18). The starting point is the most general solution, for a given set of group relations, as given by (1.29). The right hand side of the MBE will be a *consequence* (see App.A). In this larger space one then looks for points with particularly attractive properties (for example, those corresponding to $\hat{R}_c$) and, more generally, explores the consequences of the free parameters in (1.29) such as in related noncommutative geometries (Sec.3).

2 MBE for $SO_q(3)$:

We fix the normalisation by choosing the top left element (row 1, col.1) to be unity. In order to simplify the explicit form of $\hat{R}$ we denote the remaining parameters in (1.18) as follows

$$\begin{aligned} \hat{R} &= P^{(+)} + (1 + a(1+q^2))P^{(-)} + (1 + b(1+q+q^2))P^{(0)} \\ &= I + a(1+q^2)P^{(-)} + b(1+q+q^2)P^{(0)} \end{aligned} \tag{2.30}$$

The projectors are given explicitly at the end of *App.A*. The 9×9 symmetric $\hat{R}$ is now

$$\hat{R}(a,b;q) =$$

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & (1+a) & 0 & -aq & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & (1+aq+b) & 0 & (b+a(q-1))\sqrt{q} & 0 & (b-a)q & 0 & 0 \\
0 & -aq & 0 & (1+aq^2) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & (b+a(q-1))\sqrt{q} & 0 & (1+a(q-1)^2+bq) & 0 & (bq-a(q-1))\sqrt{q} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & (1+a) & 0 & -aq & 0 \\
0 & 0 & (b-a)q & 0 & (bq-a(q-1))\sqrt{q} & 0 & (1+aq+bq^2) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -aq & 0 & (1+aq^2) & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}. \qquad (2.31)$$

This $\hat{R}(a,b;q)$ satisfies the braid equation for

$$\begin{aligned}(1): a = -q^{-2}, &\quad b = -q^{-2} + q^{-3} \\ (2): a = -1, &\qquad b = -1 + q \end{aligned} \qquad (2.32)$$

the two sets giving mutually inverse matrices.

For this $\hat{R}$ (the parameters (a,b) being implicit and I being the 3×3 unit matrix) we define

$$\hat{R}_{12} = \hat{R} \otimes I, \quad \hat{R}_{23} = I \otimes \hat{R} \qquad (2.33)$$

The general srtucture of the MBE is presented in *App.A*. We present here three cases obtained for particular constraints on (a,b). One has

$$\hat{R}_{12}\hat{R}_{23}\hat{R}_{12} - \hat{R}_{23}\hat{R}_{12}\hat{R}_{23} = l_1(\hat{R}_{12} - \hat{R}_{23}) + l_2(\hat{R}_{12}^2 - \hat{R}_{23}^2) \qquad (2.34)$$

where for

*Case*1: $a = 0$ and arbitrary b,

$$l_2 = 0; \qquad l_1 = (1 + b(1 + q + q^2) + b^2q^2) \qquad (2.35)$$

*Case*2: $b = (1-q)a$

$$l_2 = (1+a); \qquad l_1 = (3 + 2(2+q^2)a + (1+2q^2)a^2) \qquad (2.36)$$

Setting $a = -1$ one obtains the case (2) of (2.32) with

$$l_1 = 0, \quad l_2 = 0$$

*Case*3: $b = (1-q^{-1})a$

$$l_2 = (1+q^2a); \qquad l_1 = (3 + 2(1+2q^2)a + q^2(2+q^2)a^2) \qquad (2.37)$$

Setting $a = -q^{-2}$ one obtains the case (1) of (2.32) with

$$l_1 = 0, \quad l_2 = 0$$

Let us note the following features:

Thr right hand side of (2.34) is linear only for $a = 0$ (Case 1). This is evidently not included in the standard cases (2.32). *Yet the braid equation is satisfied* for

$$a = 0, \qquad (b^2q^2 + b(1 + q + q^2) + 1) = 0$$

hence (when $q \neq 0$) for

$$b = -\frac{1}{2q^2}(1 + q + q^2) \pm \frac{1}{2q^2}((1 + 3q + q^2)(1 - q + q^2))^{\frac{1}{2}} \tag{2.38}$$

This gives real b for $q > 0$. (See *App.B* for further discussion.)

To complete the picture,we note that the braid matrix becomes for

$$q = 0, \quad a = 0, \quad b = -1$$

$$\hat{R}(0, -1; 0) = diag(1, 1, 0, 1, 1, 1, 1, 1, 1) \tag{2.39}$$

When a and b are independent and arbitrary even the quadratic terms of the r.h.s. of (2.34) do not suffice (*App.A*).

3 Noncommutative 3-space from $\hat{R}(a, b; q)$:

When $\hat{R}$ satisfies the braid equation (for (2.32)) the quantum vector space is discussed in [3, 4, 5]. (See in particular *Sec.*(9.3.2) of [4] and *Ex.*(4.1.22) of [5]. Our results below are to be compared to these treatments.) We will treat the more general case with parameters (a, b). The explicit form of the r.h.s. of the MBE (*Sec.*2) is not directly relevant here. We treat (a, b) as free parameters to start with. With a slight change of notation (with respect to *Sec.*1) we set in (2.30)

$$v = a(1 + q^2); \quad w = b(1 + q + q^2)$$

giving

$$\hat{R} = I + vP^{(-)} + wP^{(0)} \quad = P^{+} + (1 + v)P^{(-)} + (1 + w)P^{(0)} \tag{3.40}$$

The coordinates are denoted (x_-, x_0, x_+). Let $(x \otimes x)$ (without "tilde" for simplicity) denote the 9-component *column* obtained from the tensor product. Let (ξ_-, ξ_0, ξ_+) denote the differentials (dx_-, dx_0, dx_+). Let the columns for the other tensor products be denoted, in evident notations, as $(\xi \otimes \xi)$, $(x \otimes \xi)$, $(\xi \otimes x)$.

As in [1] we will adopt prescriptions that give commutators of (x_i, x_j) and of (ξ_i, ξ_j) independent of (v, w) while those of (x_i, ξ_j) do depend on them.

Let

$$(\hat{R} - I)(\hat{R} - (1 + w)I)(x \otimes x) = 0$$

or

$$P^{(-)}(x \otimes x) = 0 \tag{3.41}$$

This agrees with [3, 4, 5]. Now set

$$(x \otimes \xi) = M(\xi \otimes x) \tag{3.42}$$

Exterior derivation gives

$$(\xi \otimes \xi) = -M(\xi \otimes \xi) \tag{3.43}$$

or

$$(M + I)(\xi \otimes \xi) = 0$$

.

Now exterior derivation of (3.41) along with (3.42) gives the typical constraint

$$(\hat{R} - I)(\hat{R} - (1 + w)I)(M + I) = 0 \tag{3.44}$$

Hence one can choose (k being an arbitrary constant parameter)

$$(M + I) = k(\hat{R} - (1 + v)I) = k(-vP^{(+)} + (w - v)P^{(0)}) \tag{3.45}$$

From (3.43) and (3.45), due to the orthogonality of the projectors, one obtains (in agreement with [4] and [5])

$$P^{(+)}(\xi \otimes \xi) = 0, \quad P^{(0)}(\xi \otimes \xi) = 0 \tag{3.46}$$

From (3.41) and (3.46) one obtains (as in the standard treatments cited above)

$$x_- x_0 = q x_0 x_-, \qquad x_0 x_+ = q x_+ x_0$$

$$x_+ x_- - x_- x_+ = h {x_0}^2 \tag{3.47}$$

with

$$h \equiv \left(\sqrt{q} - \frac{1}{\sqrt{q}}\right)$$

and

$$\xi_-{}^2 = 0, \quad \xi_+{}^2 = 0, \quad \xi_- \xi_+ + \xi_+ \xi_- = 0$$

$$q \xi_- \xi_0 + \xi_0 \xi_- = 0, \quad q \xi_0 \xi_+ + \xi_+ \xi_0 = 0$$

$$\xi_0{}^2 = h \xi_- \xi_+ \tag{3.48}$$

Now we come to the part specific to our formalism. We define

$$\Phi_- = (\xi_- x_0 - q\xi_0 x_-), \quad \Phi_+ = (\xi_0 x_+ - q\xi_+ x_0)$$

$$\Phi_0 = (\xi_- x_+ + \sqrt{q}\xi_0 x_0 + q\xi_+ x_-), \quad \Phi_0' = (\xi_- x_+ + h\xi_0 x_0 - \xi_+ x_-) \tag{3.49}$$

Then, implementing the definition of M in terms of $\hat{R}(a,b)$ and the explicit form of the latter one obtains from (3.42), denoting

$$k_1 = -(kv+1) = -(ka(q^2+1)+1),$$

the module srtucture

$$x_-\xi_- = k_1\xi_- x_-; \quad x_-\xi_0 = k_1\xi_- x_0 + ka\Phi_-$$

$$x_-\xi_+ = k_1\xi_- x_+ + kaq\Phi_0' + kb\Phi_0; \quad x_0\xi_- = k_1\xi_0 x_- - kaq\Phi_-$$

$$x_0\xi_0 = k_1\xi_0 x_0 + ka(q-1)\sqrt{q}\Phi_0' + kb\sqrt{q}\Phi_0$$

$$x_0\xi_+ = k_1\xi_0 x_+ + ka\Phi_+; \quad x_+\xi_- = k_1\xi_+ x_- - kaq\Phi_0' + kbq\Phi_0$$

$$x_+\xi_0 = k_1\xi_+ x_0 - kaq\Phi_+; \quad x_+\xi_+ = k_1\xi_+ x_+ \tag{3.50}$$

One can verify that one obtains the relations given in $Ex.4.1.22$ of [5] (page 133) on setting

$$k = q^2, \quad a = -q^{-2}, \quad b = -q^{-2} + q^{-3} \tag{3.51}$$

For

$$k = 1, \quad a = -2(1+q^2)^{-1}, \quad b = 0 \tag{3.52}$$

one obtains the case (1.26) of $\hat{R}_{(c)}$ (with $\hat{R}^2_{(c)} = I$) where

$$P^{(-)} = \frac{1}{2}(\hat{R}_{(c)} - 1); \quad (M+I) = 2(P^{(+)} + P^{(0)}) \tag{3.53}$$

Hence (3.47) and (3.48) are conserved along with a particularly simple form of (3.50). Here one moves out of the restricted space of solutions of BE (or YBE)) to implement the particular simpicity of $\hat{R}_{(c)}$. (See the relevant remarks in $Sec.$5).)

4 MBE and Baxterization:

In [1] we briefly pointed out that MBE and Baxterization are two complementary aspects of the same basic construction: the general solution of $\hat{R}TT$ equation for a given set of group relations of the elements of T. For the cases considered in [1] (generalisable to $GL_q(N)$) the correspondance is relatively simple. In (1.2) the same, single parameter K appears in each member on the left leading to the non-zero r.h.s. (thus modifying the BE) as shown in (1.2). In a complementary approach, one can vary K in different members on the left in a prescribed fashion (indicated in [1]) so that the r.h.s. remains zero. This is

Baxterization. The *same* parameter that leads to MBE thus leads also to integrable systems in a complementary fashion.

One can make a parallel study for $SO_q(3)$ (generalisable to $SO_q(N)$ and $Sp_q(N)$). But the presence of three projectors and hence (apart from a normalisation factor) of two arbitrary parameters leads to a more complex situation. Even restricted cases give the MBE of (2.34) with quadratic terms on the right, the general structure being given in *App.A*. Let us now look at the complementary situation, namely, Baxterization.

In [4] a solution is given (p.295 − 297) of

$$\hat{R}_{12}(x)\hat{R}_{23}(z)\hat{R}_{12}(y) - \hat{R}_{23}(y)\hat{R}_{12}(z)\hat{R}_{23}(x) = 0 \tag{4.54}$$

for the restricted case where

$$z = xy \tag{4.55}$$

In our notations the solution of [4] reads

$$\hat{R}(x) = I + \frac{x-1}{q-q^{-1}}(qI-(q+q^{-1})P^{(-)}-(q-q^{-2})P^{(0)}) + \frac{x+1}{x\alpha_{(\pm)}+1}\Big(1-\frac{q^4-q^{-4}}{q-q^{-1}}\Big)P^{(0)} \tag{4.56}$$

where for $SO_q(3) \quad \alpha_{(\pm)} = \pm q^{(2\pm 1)}$.

(We have suppressed an overall factor $h(x)$ which cancels out in (4.54). We have used (81) of *p*.275 of [4] to express K of this reference by $P^{(0)}$. Finally we have written x for x^γ of [4]. One can rewrite (4.55) as $z^\gamma = x^\gamma y^\gamma$ and then redefine again absorbing γ. Our notation displays the single parameter that is effectively implemented. Adjusting the normalisation one finds a particular case of (3.40).)

The restriction (4.55) is however not essential in Baxter's criterion for commuting tansfer matrices. One can ask whether for given (x, y) a z can be found assuring (4.54). We present a relatively simple solution providing the complementary facet of the MBE of (2.35). Let

$$\hat{R}(w) = I + wP^{(0)} \tag{4.57}$$

Then

$$\hat{R}_{12}(w)\hat{R}_{23}(w')\hat{R}_{12}(w'') - \hat{R}_{23}(w'')\hat{R}_{12}(w')\hat{R}_{23}(w) = 0 \tag{4.58}$$

for

$$w' = \frac{w + w'' + ww''}{1 - q^2(1+q+q^2)^{-2}ww''} \tag{4.59}$$

We have used results in *App.A* (in particular (6.74)) to derive (4.59), but it can be verified directly. Generalisations are not evident. But a systematic study of possibilities in this context is desirable.

5 Remarks:

We conclude by noting the following points.

(1) : After the introductory remarks (*Sec.*1) on the spectral decomposition of $\hat{R}$ for vector representations of $GL_q(N)$, $SO_q(N)$ and $Sp_q(N)$, from *Sec.*2 onwards we restricted our study to $SO_q(3)$. But a substantial part of our results are evidently generalisable to $SO_q(N)$ with $N > 3$. The crucial feature is the number of projectors in the spectral decomposition. The MBE for $GL_q(2)$ with two projectors and that for $SO_q(3)$ with three exhibit major differences, made explicit here. But in $SO_q(N)$ the number of projectors does not vary with N. Still a careful stdy of of the case of $SO_q(4)$ and comparison of the results with those for $SO_q(3)$ would be of real interest. This is beyond the scope of this paper.

(2) : In [1] we started with the criterion of using the most general solution $\hat{R}$ of the $\hat{R}TT$ relations for a given set of group relations of the elements of T. This, being implemented in the standard trilinear structure of the braid equation, modified the right hand making it non-zero but *linear* in the R's as shown in (1.2). *No apriori postulate was was made concerning the r.h.s. of the equation.* The explicit form was a *consequence* of the free parameter K in $\hat{R}$. It was then noted in [1] that the MBE thus obtained (eqn.(1.2) of this paper) coincided with that introduced in [2] (and sources cited there). Now this is seen to be a *coincidence* valid for the cases studied in [1] (generalisable to $GL_q(N)$). All those cases involved two projectors (the sum being I). As soon as this number increases (such as already for $SO_q(3)$) the r.h.s. has a more complex structure. Our starting point (the general solution of $\hat{R}TT$) is exactly the same here. But only in the very particular case (2.35) one has a linear srtucture on the right.

(3) : In *Sec.*3 we present all the relations involving the coordinates and the differentials. But much remains to be done to better understand the noncommutative space thus obtained. The properties of the Φ's introduced in (3.49) deserve study. To render the geometry more transparent one should construct the "frame basis" in terms of operators commuting with the algebra. (See [5], [6] and sources cited there.) Thus equipped, one can study possible attractive consequences concerning the metric of implementing $\hat{R}_c$ as in (3.53). We hope to explore these aspects elsewhere.

(4) : After [1] we emphasise here again the complementary nature of MBE and Baxterization. We provide a simple new example of the latter in (4.57). Let us repeat another point made in [1]. The standard braid equation can be made to correspond to the third Reidemeister move in knot theory. Hence (4.54) can be viewed as a parametrisation of this move. It would be worth exploring in the context of knot theory whether this provides access to a richer class of invariants.

(5) : Here we have studied coboundary $\hat{R}$ martices in vector representations in terms of projectors. An approach using Drinfeld's transformation can be found in [7].

(6) : Finally let us recapitulate the remarkable properties of the solution provided by (2.38). It satifies the $\hat{R}TT$ equation for the standard group relations of $SO_q(3)$. It satisfies the standard braid equation ((2.34) with zero r.h.s.). It continues (as shown in *App.B*) to

be nontrivial even for $q = 1$. It can be relatively simply Baxterized as shown at the end of *Sec.*4. This sets the stage for a full study of the corresponding integrable model. This also suggests a search of analogous new solutions, more generally, for $SO_q(N)$ and $Sp_q(N)$ and also for higher dimensional representations of $SO_q(3)$. We hope to explore such possibilities elsewhere.

6 APPENDIX A : General structure of MBE

In the notation of (2.30) or (3.40)

$$\hat{R} = I + vP^{(-)} + wP^{(0)} \tag{6.60}$$

For elucidating the sructure of the consequent MBE we start with a number of definitions and auxiliary relations .

We define

$$X_1 = P^{(-)}_{(12)}, \quad X_2 = P^{(-)}_{(23)}; \quad Y_1 = P^{(0)}_{(12)}, \quad Y_2 = P^{(0)}_{(23)} \tag{6.61}$$

The orthonormal properties of the projectors imply (for $i = (1, 2)$)

$$X_i{}^2 = X_i, \quad Y_i{}^2 = Y_i; \qquad X_iY_i = Y_iX_i = 0 \tag{6.62}$$

We also define

$$S_1 = X_1 - X_2, \qquad S_2 = Y_1 - Y_2 \tag{6.63}$$

$$S_3 = (X_1X_2Y_1 + X_1Y_2X_1 + Y_1X_2X_1) - (X_2X_1Y_2 + X_2Y_1X_2 + Y_2X_1X_2) \tag{6.64}$$

$$S_4 = (Y_1Y_2X_1 + Y_1X_2Y_1 + X_1Y_2Y_1) - (Y_2Y_1X_2 + Y_2X_1Y_2 + X_2Y_1Y_2) \tag{6.65}$$

$$S_5 = (X_1X_2X_1 - X_2X_1X_2), \qquad S_6 = (Y_1Y_2Y_1 - Y_2Y_1Y_2) \tag{6.66}$$

Using these definitions and the properties (6.62) one obtains quite generally,

$$\hat{R}_{12}\hat{R}_{23}\hat{R}_{12} - \hat{R}_{23}\hat{R}_{12}\hat{R}_{23} = (v + v^2)S_1 + (w + w^2)S_2 + v^2wS_3 + vw^2S_4 + v^3S_5 + w^3S_6 \tag{6.67}$$

The S's depend on q *only*. The dependence on (v, w) are explicitly displayed in the coefficients of (6.67).

Now we exploit systematically the constraints on the S's provided by the known solutions, namely, $(2.32), (2.34), (2, 35), (2.36)$ and (2.37).

The r.h.s. of (6.67) must vanish for the braid solutions (2.32), namely, for

$$v = -(1 + q^{-2}) \quad w = -(1 - q^{-3}) \tag{6.68}$$

and also for

$$v = -(1 + q^{2}) \quad w = -(1 - q^{3}) \tag{6.69}$$

Implementing these (for $q \neq 1$) we choose to express (S_3, S_4) as

$$S_3 = \frac{q^2}{(q^2+1)(q^2+q+1)}S_1 - \frac{q^2}{(q^2+1)^2}S_2 - \frac{(q^2+1)}{(q^2+q+1)}S_5 - \frac{(q^3-1)^2}{q(q^2+1)^2}S_6 \tag{6.70}$$

$$S_4 = \frac{q^3}{(q^3-1)^2}S_1 + \frac{q^2}{(q^2+1)(q^2+q+1)}S_2 - q\frac{(q^2+1)^2}{(q^3-1)^2}S_5 + \frac{(q^3-1)(q-1)}{q(q^2+1)}S_6 \tag{6.71}$$

Now note that implementing (6.62) one can express the r.h.s. of (2.34) as

$$l_1(\hat{R}_{12} - \hat{R}_{23}) + l_2(\hat{R}_{12}^2 - \hat{R}_{23}^2) = (l_1 v + l_2(2v + v^2))S_1 + (l_1 w + l_2(2w + w^2))S_2 \tag{6.72}$$

Combining this result with $(Case.1)$ or (2.35) one obtains (since $v = 0$, $l_2 = 0$)

$$\begin{aligned} l_1 w S_2 &= \Big((w + w^2) + \frac{q^2 w^3}{(q^2+q+1)^2}\Big)S_2 \\ &= (w + w^2)S_2 + w^3 S_6 \end{aligned} \tag{6.73}$$

Hence

$$S_6 = q^2(q^2+q+1)^{-2}S_2 \tag{6.74}$$

From (6.70), (6.71) and (6.74) one obtains

$$S_3 = \frac{q^2}{(q^2+1)(q^2+q+1)}S_1 - \frac{q(q^2-q+1)}{(q^2+1)^2}S_2 - \frac{(q^2+1)}{(q^2+q+1)}S_5 \tag{6.75}$$

$$S_4 = \frac{q^3}{(q^3-1)^2}S_1 + \frac{q(q^2-q+1)}{(q^2+1)(q^2+q+1)}S_2 - q\frac{(q^2+1)^2}{(q^3-1)^2}S_5 \tag{6.76}$$

Combining all the preceding results one finally obtains

$$\begin{aligned} \hat{R}_{12}\hat{R}_{23}\hat{R}_{12} - \hat{R}_{23}\hat{R}_{12}\hat{R}_{23} &= c_1 S_1 + c_2 S_2 + c_5 S_5 \\ &= c_1(P^{(-)}_{(12)} - P^{(-)}_{(23)}) + c_2(P^{(0)}_{(12)} - P^{(0)}_{(23)}) \end{aligned} \tag{6.77}$$

$$+c_5(P^{(-)}_{(12)}P^{(-)}_{(23)}P^{(-)}_{(12)} - P^{(-)}_{(23)}P^{(-)}_{(12)}P^{(-)}_{(23)})$$

Here

$$c_1 = v + v^2 + \frac{v^2wq^2}{(q^2+1)(q^2+q+1)} + vw^2\frac{q^3}{(q^3-1)^2}$$

$$c_2 = w + w^2 + \frac{w^2vq(q^2-q+1)}{(q^2+1)(q^2+q+1)} - wv^2\frac{q(q^2-q+1)}{(q^2+1)^2} + \frac{w^3q^2}{(q^2+q+1)^2}$$

$$c_5 = \frac{v}{(q^3-1)^2}\Big((q^3-1)v + (q^2+1)w)\Big)\Big((q^3-1)v - q(q^2+1)w\Big) \tag{6.78}$$

As checks one verifies that $c_1 = c_2 = c_5 = 0$ for (6.68) and (6.69). Moreover, $c_5 = 0$ for

$$(q^2+1)w = -(q^3-1)v$$

and

$$q(q^2+1)w = (q^3-1)v$$

Thus one gets back, respectively, $Case2$ of (2.36) and $Case3$ of (2.37).

The form (6.77) makes the dependence on (v,w) entirely explicit, the c's being given by (6.78) and the S's depending only on q. This is particularly suitable for our purpose. The P's can be reexpressed in terms of $\hat{R}(v,w)$ using (1.20). But only for $c_5 = 0$ (cases $(1,2,3)$ of $(2.35), (2.36), (2.37)$ respectively) one obtains a relatively simple form as in (2.34). For S_5 there is no crucial simplification as for S_6 in (6.74). Setting $w = 0$ in (6.77) one gets no new simplification but an identity.

Throughout this paper the projectors $P^{(-)}$ and $P^{(0)}$ have served as the essential building blocks. They can be easily extracted comparing (2.30) and (2.31). But for completeness and convenience they are presented below explicitly.

$$(q^2+1)P^{(-)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -q & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q & 0 & (q-1)\sqrt{q} & 0 & -q & 0 & 0 \\ 0 & -q & 0 & q^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & (q-1)\sqrt{q} & 0 & (q-1)^2 & 0 & -(q-1))\sqrt{q} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -q & 0 \\ 0 & 0 & -q & 0 & -(q-1)\sqrt{q} & 0 & q & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -q & 0 & q^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{6.79}$$

$$
(q^2+q+1)P^{(0)} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & \sqrt{q} & 0 & q & 0 & 0 \\
0 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \sqrt{q} & 0 & q & 0 & q\sqrt{q} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & q & 0 & q\sqrt{q} & 0 & q^2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}. \tag{6.80}
$$

7 APPENDIX B: Nontrivial BE for q=1

We pointed out in *Sec.*2 that for $a = 0$ and b satisfying (2.38) one obtains two solutions of BE (not modified, with vanishing r.h.s.). Setting

$$
q = 1
$$

in (2.38) one obtains

$$
b^2 + 3b + 1 = 0
$$

or

$$
b = \frac{1}{2}(-3 \pm \sqrt{5}) \equiv -e^{\mp m} \tag{7.81}
$$

(It is amusing to note the relation of b, or rather that of $-(b+1)$ with the famous Golden Mean i.e. $\frac{1}{2}(\sqrt{5}-1)$.)

Now one obtains

$$
\hat{R}(\mp m) = I - 3e^{\mp m}\hat{P}^{(0)} \tag{7.82}
$$

Where the projector $\hat{P}^{(0)}$ is obtained from $P^{(0)}$ by setting $q = 1$. Thus

$$
3\hat{P}^{(0)} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
$$

From (7.82) one easily obtains the nontrivial Hecke condition

$$(\hat{R}(\mp m) - I)(\hat{R}(\mp m) + (3e^{\mp m} - 1)I) = 0 \tag{7.83}$$

Thus $\hat{R}(\mp m)$ are *not* coboundary matrices. They cannot be obtained by twisting I since

$$\left(\hat{R}(\mp m)\right)^2 \neq I$$

For

$$q = 1, \quad b = 0, \quad a = -1 \tag{7.84}$$

again (from (2.30), (2.36), (2.37)) one obtains $\hat{R}$ satifying BE given by

$$\hat{R} = I - 2\hat{P}^{(-)} \tag{7.85}$$

where $\hat{P}^{(-)}$ is obtained from $P^{(-)}$ setting $q = 1$ in (6.79).

But now, in contrast to (7.83), one has

$$\hat{R}^2 = I$$

The $R(\mp m)$ satisfying YB (Yang-Baxter equation) can be obtained from (7.82) as

$$R(\mp m) = P\hat{R}(\mp m) = P - 3e^{\mp m}\hat{P}^{(0)} \tag{7.86}$$

where the matrix P (permuting the rows $(2,4),(3,7),(6,8)$) leaves $\hat{P}^{(0)}$ invariant.

In Hieterinta's classification [8] of 4×4 R matrices appear examples without free parameters. Such a case has been studied [9] in the context of "exotic bialgebras". Here we have obtained 9×9 examples of such matrices.

I discussed many aspects of this paper with M.A.Sokolov. He also provided, using a program, the explicit expressions of the coefficients in the solutions in *Sec.*2. It is a pleasure to thank him . I also thank John Madore for instructive discussions concerning the noncommutative geometric aspects.

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